Analytical estimate of the critical velocity for vortex pair creation in trapped Bose condensates

M. Crescimanno
Department of Physics and Astronomy, Youngstown State University, Youngstown, Ohio 44555

C. G. Koay and R. Peterson
Physics Department, Berea College, Berea, Kentucky 40404

R. Walsworth
Harvard-Smithsonian Center for Astrophysics, 60 Garden Street, Cambridge, Massachusetts 02138

(Received 18 January 2000; published 15 November 2000)

We use a Thomas-Fermi approximation that includes the leading kinetic terms due to fluid motion to estimate analytically the critical velocity for the formation of vortex pairs in harmonically trapped Bose-Einstein condensates. We find rough agreement between this analytical estimate and recent experiments on trapped sodium gas condensates.

PACS number(s): 03.75.Fi, 67.40.Vs, 67.57.De

I. INTRODUCTION

The experimental realization of Bose-Einstein condensation in cold trapped gas has stimulated great interest in weakly interacting inhomogeneous quantum fluids. The gaseous condensates are excellent systems for the study of quantum fluids, because (i) they are dilute, and hence admit an accurate description by mean-field theory; (ii) system parameters such as density, temperature, and trapping potential are under fine experimental control; and (iii) sensitive diagnostics (mostly optical) have been developed to probe condensate behavior. In particular, gaseous condensates may serve as a powerful laboratory to study long established questions about the breakdown of superfluidity as well as newer questions about the effects of spatial inhomogeneity.

It is well known theoretically that collective excitations in a weakly interacting Bose-Einstein condensate (BEC) determine a superfluid critical velocity, below which the condensate flow relative to a perturbing object or potential is without drag. This critical velocity is given by the Landau criterion [1]

$$v_c = \min\left(\frac{E(I)}{I}\right), \tag{1.1}$$

where $E(I)$ is the energy of a condensate collective excitation of linear momentum (or impulse) I. For liquid ^{4}He, the breakdown of superfluidity is observed experimentally to occur at a critical velocity that is much smaller than that due to the excitation of phonons or rotons [2]. Beginning with Feynman more than 40 years ago [3], it has long been thought that the critical velocity of superfluid ^{4}He is set by the creation of complex vortex structures (e.g., pairs, rings, and loops) depending on system geometry, temperature, etc. However, it has not been possible to fully verify this basic hypothesis because of the strong interactions in a liquid, which complicate quantitative comparison of superfluid ^{4}He experiments [4] with theory [5], as well as the limited ability to vary the liquid density. Dilute gas BEC may provide an experimental system to test quantitatively the link between the superfluid critical velocity and the creation of vortices.

In landmark recent experiments [6–8], Ketterle and coworkers at MIT measured a critical velocity for the excitation of a sodium gas BEC when a perturbing potential (a blue-detuned laser beam) was moved through the trapped quantum fluid. These measurements agree qualitatively with both the well-known analytical calculation for a homogeneous weakly interacting BEC in a channel of finite diameter [2,3,9] and recent numerical calculations [10–16] based on the Gross-Pitaevskii equation [17]. The MIT critical velocity experiments have also stimulated analytical investigations of vortex nucleation [18] and the effect of an elongated condensate geometry [19].

Detailed understanding of the breakdown of superfluidity in Bose condensates will likely require further experimentation (e.g., to vary the trapped BEC geometry and density, and to create greater spatial symmetry in the BEC perturbation), as well as additional theoretical work—both numerical and analytical. To this end, we report in this paper an analytical calculation of the critical velocity for vortex pair production in a harmonically trapped dilute Bose condensate. We employ a Thomas-Fermi (TF) approximation to the Bogoliubov mean-field theory [20], treating the BEC as a fluid with an exotic equation of state. We include leading kinetic energy terms caused by the fluid motions of two counter-rotating (but pinned) vortices, and use these kinetic energy terms as an effective potential for the BEC. We then compute, approximately, the energy $E(I)$ and impulse I of the vortex pair. Using these in Eq. (1) and trap characteristics from the most recent MIT experiments [7,8], we find reasonable quantitative agreement between our analytical estimate of v_c and the experimentally measured value.

As a step toward checking the analytical calculation method for BEC vortices, we also compute the Bogoliubov many-body ''wave function'' (ψ) for a condensate with a single central vortex, and then calculate E, I, and v_c for this case. Since our calculations are based on an effective hydrodynamic model of BEC, the vortices that we consider do not, strictly speaking, have quantized angular momentum in units of $\hbar$. However, we find that this use of the TF approximation agrees quite well with a vortex solution computed numerically.

We believe the advantages of the analytical method presented here are (i) its simplicity, and (ii) that it works well for the limit of large interparticle interaction, which is the limit largely being explored by current experiments. In addition, this Thomas-Fermi analysis suggests the manner in which v_c/c_B decreases as the interparticle interaction increases (where c_B is the speed of sound at the trap center).

In Sec. II below we introduce both the model and the Thomas-Fermi approximation used in our calculation. In Sec. III we apply this model and approximation to the creation of vortex pairs near the center of a harmonically trapped dilute BEC, and then compare the analytically estimated critical velocity with the recent measurements by the MIT group [6–8]. In Sec. IV we show that our analytical calculation of v_c for a single central vortex in a trapped BEC agrees well with numerical simulation. Finally, in Sec. V we summarize the implications of this work.

II. MODEL OF TRAPPED BEC AND THE THOMAS-FERMI LIMIT

We restrict our discussion to the dynamics of a harmonically trapped single-component dilute Bose condensate at $T=0$, which is well described by the Gross-Pitaevskii (GP) equation [17]

$$-\frac{\hbar^2}{2M}\nabla^2\psi' + \frac{KR^2}{2}\psi' + U|\psi'|^2\psi' = \mu'\psi', \qquad (2.1)$$

where the field ψ', the diagonal part of the multiparticle wave function, serves as the condensate order parameter, and is referred to as ''the wave function of the condensate''; M is the atomic mass; μ' is the BEC chemical potential; and $U|\psi'|^2$ is an effective nonlinear potential arising from the atomic interactions (due to s-wave binary atomic collisions). Here $U=(4\pi\hbar^2 a_s)/M$, where a_s is the atomic s-wave scattering length. For simplicity, we assume the trap has cylindrical symmetry and take $R^2=x^2+y^2=|\vec{r}|^2$, with force constant K and ignore the trapping potential along the symmetry ($\hat{z}$) axis. Note that we solve the above GP equation subject to the constraint that the total number of atoms N is fixed,

$$N=\int d^2r\, dz|\psi'|^2. \qquad (2.2)$$

For ease of calculation we use scaled harmonic-oscillator units (h.o.u.) in which the units of length, time, and energy are $\sqrt{\hbar/2M\omega}$, $1/2\omega$, and $\hbar\omega$, respectively, where $\omega=\sqrt{K/M}$ is the trap angular frequency. Also, we reduce the GP equation to two dimensions by assuming the wave function is separable: $\psi'=\psi(\vec{r})\Phi(z)$. Thus, adopting the notation of Jackson *et al.* [12], we find

$$-\nabla^2\psi + \frac{R^2}{4}\psi + C|\psi|^2\psi = \mu\psi, \qquad (2.3)$$

where we have scaled ψ so that

$$1=\int d^2r|\psi|^2. \qquad (2.4)$$

Solving Eq. (2.3) subject to Eq. (2.4) fixes μ, the dimensionless chemical potential. The dimensionless interaction parameter is

$$C=\frac{2MUN}{\hbar^2 a_t \zeta}, \qquad (2.5)$$

where

$$a_t=\left(\frac{\hbar}{2M\omega}\right)^{1/2} \qquad (2.6)$$

is the classical turning-point width of the condensate in the harmonic trap in the $C\to 0$ limit (equal to the unit of length in h.o.u.), and ζ is the trap aspect ratio given approximately by $\zeta=(\int dz\, z\Phi^*\Phi)/a_t|\Phi(0)|^2$.

For large C—which corresponds to most current experiments—the solution to the GP equation without vortices is well approximated by the Thomas-Fermi (TF) limit [20]. In this limit one neglects the derivative term in Eq. (2.3), yielding

$$\psi(R)=\left(\frac{\mu-R^2/4}{C}\right)^{1/2} \quad \text{and} \quad \mu=\left(\frac{C}{2\pi}\right)^{1/2}, \qquad (2.7)$$

where the second relation comes from the normalization condition, Eq. (2.4). [See also Eq. (3.4) and discussion below.] Equations (2.7) will be used extensively in this paper as the no-vortex solution that we compare with the vortex solutions. For reference, note that in the TF limit (i.e., the long-wavelength limit), the local speed of (Bogoliubov) sound in the condensate is given in h.o.u. by $c_B=\sqrt{2C|\psi|^2}$ (see, for example, Ref. [12]).

The many-body wave function of a vortex in the condensate will have a spatially dependent phase [see Eq. (3.1) below] and a spatially dependent amplitude, which we refer to as the ''envelope function.'' Let $\phi(\vec{r})$ and $A(\vec{r})$ be the phase and envelope functions (both real) parametrizing this wave function via $\psi=Ae^{i\phi}$. Using this parametrization in Eq. (2.3), and equating real and imaginary parts, we find

$$-A\nabla^2 A + A(\vec{\nabla}\phi)^2 + \frac{R^2}{4}A + CA^3 = \mu A, \qquad (2.8)$$

$$-A\nabla^2\phi - 2\vec{\nabla}A\cdot\vec{\nabla}\phi = 0. \qquad (2.9)$$

One may think of Eq. (2.8) as an equation of hydrostatic equilibrium and Eq. (2.9) as a continuity equation. Our use of the TF approximation for BEC with vortices consists of first solving the continuity equation [Eq. (2.9)] for $\phi(\vec{r})$, and then using that solution in Eq. (2.8) to determine $A(\vec{r})$, neglecting the $\nabla^2 A$ term. Thus, in this approximation the envelope function $A(\vec{r})$ is the solution of a simple algebraic equation.

Once we have normalized the vortex wave function via Eq. (2.4), we compute the energy and integrated impulse. The energy functional is

$$E=\int d^2r\left(\vec{\nabla}\psi^*\cdot\vec{\nabla}\psi+\frac{R^2}{4}\psi^*\psi+\frac{C}{2}(\psi^*\psi)^2\right). \tag{2.10}$$

Using $\psi=Ae^{i\phi}$ with A and ϕ real, one finds

$$E=\int d^2r\left((\vec{\nabla}A)^2+(A\vec{\nabla}\phi)^2+\frac{R^2A^2}{4}+\frac{C}{2}A^4\right). \tag{2.11}$$

The local momentum density is $\vec{P}=i\hbar/2[(\vec{\nabla}\psi^*)\psi-\psi^*\vec{\nabla}\psi]$, which we use to compute the total impulse:

$$I=\int d^2r|\vec{P}|=\hbar\int d^2r\,A^2|\vec{\nabla}\phi|. \tag{2.12}$$

We can then estimate the critical velocity for vortex pair creation by applying the Landau criterion [Eq. (1.1)],

$$v_c=\frac{E_1-E_0}{I_1-I_0}, \tag{2.13}$$

where the indices ''1'' and ''0'' refer to whether a pair of unit charged vortices is present or absent, respectively. In the next section we describe the details of an analytical computation of this ratio for a vortex pair near the center of the trap.

III. ANALYTICAL CALCULATION OF THE VORTEX PAIR CRITICAL VELOCITY

We consider a dilute BEC containing a pair of counter-rotating vortices with opposite charge or vorticity $\pm n$. The vortices are assumed to be located symmetrically about the trap center, with the cores a distance d apart. We apply the TF approximation to the hydrostatic equilibrium expression [Eq. (2.8)] by assuming that ∇^2A is negligible. In addition, we adopt the ansatz that the two vortices are far enough from each other, but near enough to the trap center, that the total phase advance about the vortex pair is the simple sum of the phase advances of the two vortices viewed individually. This ansatz is akin to the analytical approximation developed by Feynman [3] for homogeneous BEC, and also used by Fetter in a similar context [21]. Practically, the ansatz is equivalent to requiring $1/\mu<d^2<\mu$ in dimensionless h.o.u. Operationally it means that the wave function's phase $\phi(\vec{r})$ satisfies $\nabla^2\phi=0$ everywhere outside the vortex cores, which implies from the continuity expression [Eq. (2.9)] that $\vec{\nabla}A\cdot\vec{\nabla}\phi$ must be zero in this region. Our solution for $A(\vec{r})$ in this case satisfies the continuity equation near the vortex cores, and also far away from the vortices, since radial gradients of the phase vanish as $1/R^2$. In addition, the spatial integral of the continuity equation vanishes identically, and deviations due to the approximations described above are asymptotically bounded.

We have found that neglect of the ∇^2A term in Eq. (2.8) leads to the main systematic errors in the analytical calculation described here. However, the deviations are small for parameters typical of current experiments.

With the approach outlined above, the ansatz we take for the phase of the vortex pair wave function is

$$\phi(\vec{r})=n\arctan\left[\frac{\sin\theta-\dfrac{d}{2R}}{\cos\theta}\right]-n\arctan\left[\frac{\sin\theta+\dfrac{d}{2R}}{\cos\theta}\right], \tag{3.1}$$

where R and θ are the usual polar coordinates as measured from the trap center. Employing the TF approximation as described above, we find the condensate wave-function envelope to be

$$A^2(\vec{r})=\frac{1}{C}\left(\mu-\frac{R^2}{4}-\frac{d^2n^2}{R^2d^2\cos^2\theta+(R^2-d^2/4)^2}\right), \tag{3.2}$$

where the last term is the leading kinetic energy contribution, due to the $|\vec{\nabla}\phi|^2$ term in Eq. (2.8). Note that this kinetic energy contribution is simply $1/(r_1r_2)^2$, where r_1 and r_2 are the respective distances measured from the vortex cores to the point (R,θ).

The above calculation scheme for $A(\vec{r})$ breaks down very close to the vortex cores (small r_1 or r_2) and also at large R, since the right-hand side of Eq. (3.2) becomes negative. Excluding the vortex core regions from the domain complicates the analytic evaluation of energy and impulse [Eqs. (2.11) and (2.12)] precisely where the TF approximation fails. To circumvent this calculational difficulty, we adopt a regulated expression for $A(\vec{r})$:

$$A^2(\vec{r})=\frac{1}{C}\left(\mu-\frac{R^2}{4}-\frac{d^2n^2}{R^2d^2\cos^2\theta+(R^2-d^2/4)^2+\epsilon}\right), \tag{3.3}$$

where $\epsilon=(d^2n^2)/\mu$. This regulated expression is justified by the observation that for vortex pairs not too far from the trap center ($d^2<\mu$), the contribution to $A(\vec{r})$ from the kinetic energy term $|\vec{\nabla}\phi|^2$ is never larger than μ. The regulation markedly affects the wave-function envelope near the vortex cores, but has a small effect on the calculated energy, impulse, and critical velocities. (This is shown for a trap-centered, single vortex in Sec. IV below.) In sum, the use of the regulated expression for $A(\vec{r})$ is an important practical step in our analytical calculation, because it allows us to perform the spatial integrals for E and I over the entire plane, rather than excluding regions near the vortex cores.

We begin the integrations by normalizing the condensate wave function for the cases of no vortices and a single vortex pair. Referring to Eq. (2.7), the normalization for the no vortex case yields

$$\mu\equiv\mu_0=\left(\frac{C}{2\pi}\right)^{1/2}, \tag{3.4}$$

where we have performed the integration in Eq. (2.4) out to a maximum radius, $R_{\mathrm{TF}}=2\sqrt{\mu}$, i.e., the Thomas-Fermi condensate edge (the radius at which the condensate chemical potential is dominated by the trap potential in the Thomas-Fermi limit). Similarly, inserting Eq. (3.3) into Eq. (2.4), and noting that $|\psi|^2=A^2$, we determine the normalization condition for the vortex pair wave function:

$$\frac{C}{2\pi}=\mu^2-n^2\ln\left(\frac{1+\dfrac{\mu d^2}{n^2}}{1+\dfrac{d^2}{8\mu}}\right)+\cdots, \tag{3.5}$$

where we have again performed the spatial integration out to the Thomas-Fermi condensate edge, which for large C is $R_{\mathrm{TF}}=2\sqrt{\mu}-(n^2d^2)/16\mu^{5/2}+\cdots$. Note that Eq. (3.5) indicates that μ increases with the vortex charge, n, as expected. (See Ref. [22] for details of the calculation of these normalization conditions.)

Next, we compute the condensate energy. For the case of no vortex, we insert Eq. (2.7) into Eq. (2.10), and find

$$E_0=\frac{4\pi\mu_0^3}{3C}. \tag{3.6}$$

For a vortex pair of charge $\pm n$, we insert into Eq. (2.11) both the vortex pair's phase given in Eq. (3.1) and the regulated expression for $A(\vec{r})$ given in Eq. (3.3). Performing the spatial integration again out to R_{TF} (see [22] for details), we find the energy for a vortex pair to be

$$E_n=\frac{\pi}{2C}\left(\mu^2R_{\mathrm{TF}}^2-\frac{R_{\mathrm{TF}}^6}{48}\right)-\frac{d^2n^2\pi}{8C}\left[\frac{8\mu^2}{1+d^2\mu/n^2}+\ln\left(\frac{R_{\mathrm{TF}}^4\mu^2(1+d^2/2R_{\mathrm{TF}}^2)}{n^4(1+d^2\mu/n^2)}\right)\right]+\cdots. \tag{3.7}$$

Using Eqs. (3.6) and (3.7), we compute the energy difference between a trapped condensate with a symmetric pair of unit-charged, counter-rotating $(n=1)$ vortices, and the no-vortex ground state $(n=0)$. To leading order in the limit of large C, we find

$$E_1-E_0=\frac{2\pi\mu_0}{C}\ln(d^2\mu_0+1)+\cdots. \tag{3.8}$$

Note that the energy difference vanishes in the $d\to 0$ limit, as expected. Also, the energy difference varies as the natural logarithm of the distance between the vortices (at large separation), which resembles results from earlier studies of vortices in homogeneous BEC [2].

Next, we calculate the impulse I_n for the condensate with a symmetric pair of counter-rotating vortices with charge $\pm n$. (With no vortices, the trapped condensate impulse is zero.) Using Eq. (3.1), we find

$$|\vec{\nabla}\phi|=\frac{nd}{[R^2d^2\cos^2\phi+(R^2-d^2/4)^2]^{1/2}}. \tag{3.9}$$

We insert this formula and the regulated expression for $A^2(\vec{r})$ [Eq. (3.3)] into Eq. (2.12) to determine the vortex pair impulse. We find that the largest contribution to the spatial impulse integral comes at large R, and is thus dependent on the condensate size (set roughly by the Thomas-Fermi condensate edge, R_{TF}) as well as the vortex pair charge magnitude (n) and separation (d). Evaluating the integral out to R_{TF}, and assuming large C, we find the vortex pair impulse to be

$$I_n=\frac{nd\mu_0\pi}{C}\ln\left(\frac{64\mu_0}{ed^2}\right)+\cdots \tag{3.10}$$

to leading order (see [22] for details). Note that like the energy, the impulse for vortex pair creation vanishes in the small d limit, and is proportional to the vortex charge, as expected. Also, since we are using a macroscopic hydrodynamic model in our calculations, the quantization of angular momentum of the BEC wave function does not lead to a simple quantization condition on the integrated impulse.

Assembling the results for the energy and impulse leads directly to an estimate of the critical velocity v_c at which symmetrically placed, oppositely charged vortex filaments, at relative separation d, can form near the center of a Bose condensate trapped harmonically in two dimensions. In the large C limit by using Eq. (2.13), we find, in harmonic oscillator units (hou),

$$v_c\simeq\frac{2}{d}\,\frac{\ln(d^2\mu_0)}{\ln\left(\dfrac{64\mu_0}{ed^2}\right)}, \tag{3.11}$$

where again $\mu_0=\sqrt{C/2\pi}$. Recall that the derivation of Eq. (3.11) assumes that $1/\mu<d^2<\mu$ in dimensionless hou. The first inequality is the requirement that the vortices be at least several healing lengths away from each other, and the second inequality is the requirement that they be not too far from the trap center.

The expression Eq. (3.11) indicates a dependence (via μ_0 in the logarithms) of v_c on the BEC nonlinear self-energy and hence on the speed of sound, c_B. Converting Eq. (3.11) from harmonic oscillator units to physical units, we find

$$\frac{v_c}{c_B}=\frac{1}{d\sqrt{\pi a_s n}}\,\frac{\ln(2\pi d^2a_sn)}{\ln\left(\dfrac{32\pi\hbar^2a_sn}{eM^2\omega^2d^2}\right)}. \tag{3.12}$$

We now compare the prediction of Eq. (3.12) with the recent results of Ketterle *et al.* [6–8]. The onset of energy dissipation in the MIT condensate was observed at a critical velocity when moving a repulsive barrier (a blue-detuned laser beam) through the middle of the cold atom cloud. The laser beam was directed along the smaller, radial direction of the cigar-shaped condensate. The most recent experiments' [7,8] condensates had Thomas-Fermi diameters of about 65 and 130 μm in the radial and axial directions, respectively, corresponding to trap frequencies of 40 Hz in the radial direction and 20 Hz in the axial direction. The laser barrier was moved back and forth along the condensate's axial direction (perpendicular to the axis of the laser beam) at a constant speed, and energy dissipation was determined from measured changes in the condensate fraction and distortions in the condensate density distribution. Thus, the experiment

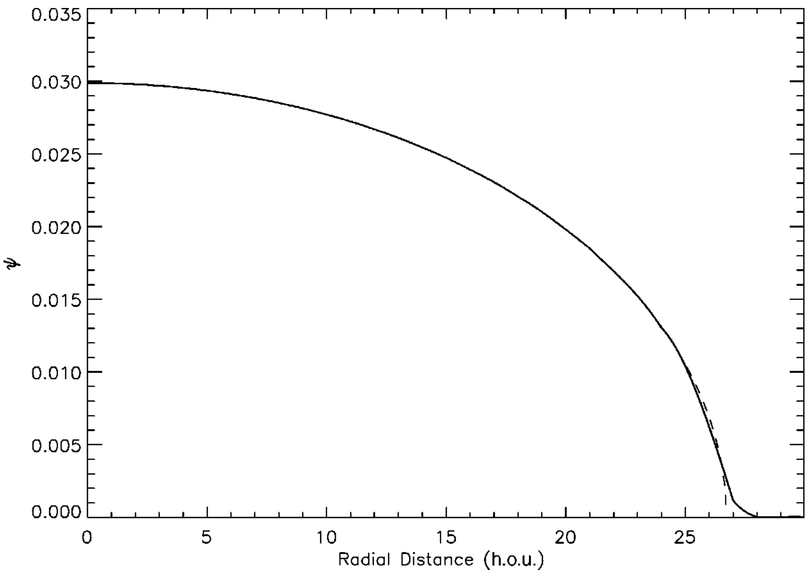

FIG. 1. Envelope function $A(R)$ for a condensate with no vortex $(n=0)$. Numerical (solid) and TF analytical (dashed) solutions are nearly identical on the scale of this graph.

was explicitly a three-dimensional system with anisotropic trapping in the plane perpendicular to the laser beam axis.

Ignoring the different geometry of our two-dimensional, isotropically trapped BEC model, we use the analytical calculation outlined above to estimate the critical velocity for vortex pair creation in the MIT condensate. We assume the vortex cores are parallel to the axis of the blue-detuned laser beam, and are separated by a distance d equal to the laser beam diameter (10 μm). In the central region of the MIT condensate [7], the atomic number density ranged from $0.9-1.9\times10^{14}$ cm^{-3}. Using 2.9 nm for the sodium s-wave scattering length (a_s), Eq. (2.5) indicates that $C=8\pi a_s n_z$ ranges between 10 000 and 50 000 (in hou) for the MIT BEC experiments. Here n_z is the total linear number density of atoms in the trap parallel to the laser beam. Thus, for such large C values we expect the TF approximation to be useful.

Parameters of the most recent MIT experiments [7,8] applied to Eq. (3.12) indicate that the the onset of vortex pair formation, and hence energy dissipation, occurs at a critical velocity v_c of $0.15c_B$ to $0.11c_B$ over the quoted range of central densities. This is in rough quantitative agreement with the results of the MIT experiments [7,8] and numerical solutions [14,16] of the Gross–Pitaevskii equation as it relates to the formation of vortex structures. We defer further interpretation of this estimate of v_c to the conclusion.

IV. A TEST OF THE ANALYTICAL CALCULATION

As a test of the analytical calculation of vortex critical velocity using a TF approximation that retains leading kinetic terms, we consider a simple, symmetric example: a single vortex of charge (i.e., vorticity) n at the center of an isotropic BEC that is harmonically trapped in two dimensions. In this case, the cylindrical symmetry of the system requires the phase of the condensate to advance proportional to the polar angle θ, and hence Eq. (2.9) limits the envelope function $A(\vec{r})$ to be a function of $R=|\vec{r}|$ alone. Also, the single-valuedness of the condensate wave function constrains the phase to be $\phi=n\theta$, where n is an integer (the vortex charge). Therefore, Eq. (2.8) reduces to a second-order ordinary differential equation for $A(R)$ that can be integrated numerically, *without making any Thomas-Fermi approximation*. This numerical calculation can then be used to test the TF analytical calculation. We employed an adaptive mesh relaxation algorithm to perform the numerical integration of Eq. (2.8) for both $n=0$ and 1, using the value $C=200\,000$ for the coefficient of the nonlinear term throughout. The computed $A(R)$ for no vortex $(n=0)$ is found to be essentially identical to the analytical result, Eq. (2.7), as shown in Fig. 1.

The TF analytical envelope function for a single central vortex of charge n is found by solving Eqs. (2.8) and (2.9) while dropping the $\nabla^2 A$ term (see also Ref. [23]). We find

$$A(R)=\left(\frac{\mu-\dfrac{R^2}{4}-\dfrac{n^2}{R^2}}{C}\right)^{1/2}, \quad \phi(\theta)=n\theta. \tag{4.1}$$

Figure 2 provides a comparison of the numerical and analytical solutions for $A(R)$ with $n=1$. Equation (4.1) shows that including the leading kinetic energy term due to the vortex in the analytical calculation is equivalent to an effective potential barrier near the vortex core caused by the angular momentum of the condensate. In this approximation, the condensate extends over the annulus $R_0<R<R_{\max}$, outside of which A vanishes. From Eq. (4.1) we see that

$$R_0^2=2\mu-2\sqrt{\mu^2-n^2}, \quad R_{\max}^2=2\mu+2\sqrt{\mu^2-n^2}. \tag{4.2}$$

Hence in this TF limit, the vortex core radius (in h.o.u.) is effectively $n/\sqrt{\mu}$ at large μ. On general grounds, we expect the radius of the vortex core to be of the order of the condensate healing length, $\xi=1/(8\pi a_s n_0)$, where a_s is the s-wave scattering length and n_0 is the local density. Thus this type of Thomas-Fermi limit reproduces the expected scaling of vortex core size, since $\xi\sim1/\sqrt{\mu}$ in dimensionless units. Note also that at large R (ignoring the trap potential) the effect of the vortex on the condensate envelope falls off as

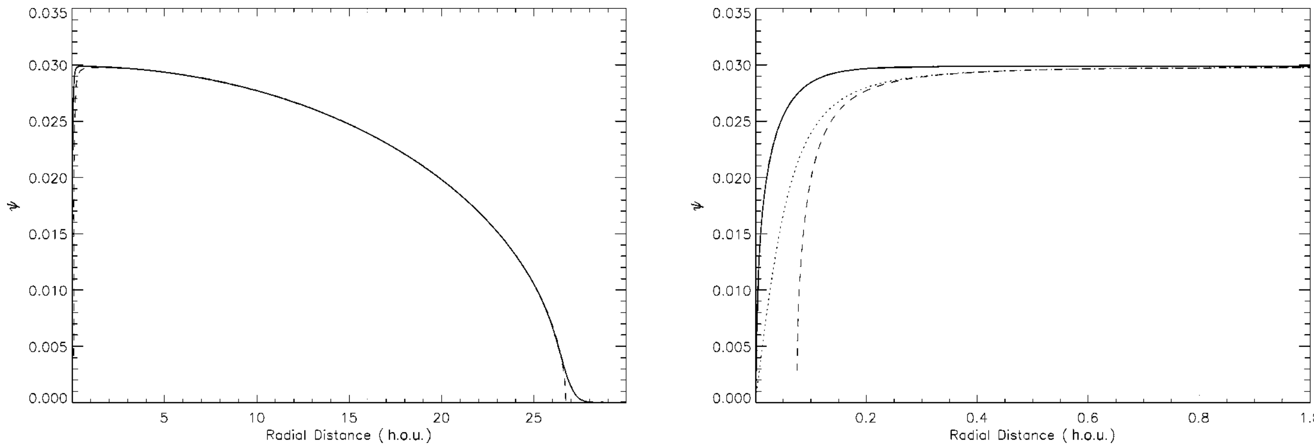

FIG. 2. Comparison of numerical and analytical envelope functions $A(R)$ for a condensate with a single central vortex of unit charge $(n=1)$. (a) Full radial extent of $A(R)$. Solid line is the numerical result and dashed line is the TF analytical calculation. (b) $A(R)$ near the trap center. Solid line is the numerical result, dashed line is the TF analytical calculation without regulation, and dotted line is with regulation.

$1/R^2$ in the analytical result, which is identical to the asymptotic behavior of the trial vortex wave function in homogeneous BEC used by Fetter [21].

The analytical calculation of the energy of a single central vortex of charge n is found by using Eq. (4.1) in Eq. (2.11) [ignoring the $(\vec{\nabla}A)^2$ term] and integrating over the annulus given in Eq. (4.2):

$$E_n=\frac{4\pi(\mu^2-n^2)^{3/2}}{3C}. \tag{4.3}$$

[Compare with Eq. (3.6).] Although not obvious from this equation, E_n increases as a function of n due to the dependence of μ on n through the normalization condition $\int d^2r|\psi|^2=1$. In particular, we find using Eq. (4.1) that normalization requires

$$\frac{C}{2\pi}=\mu\sqrt{\mu^2-n^2}-\frac{n^2}{2}\ln\left(\frac{\mu+\sqrt{\mu^2-n^2}}{\mu-\sqrt{\mu^2-n^2}}\right), \tag{4.4}$$

hence causing $R_{\max}$ [Eq. (4.2)] to increase with n. Next, using Eq. (4.1) in the analytical calculation of the impulse [Eq. (2.12)] of a single central vortex, we find

$$I_n=\frac{8\pi n}{3C}(\mu-n)^{3/2}. \tag{4.5}$$

For the no-vortex case $(n=0)$ and $C=200\,000$, numerical integration of Eq. (2.3) yields the total condensate energy: $E_0=118.965$. (All numbers here are in h.o.u.) With the analytical method we find $E_0=4\pi\mu_0^3/3C=118.94$, where $\mu_0=\sqrt{C/2\pi}=178.4$. For a single central vortex with $n=1$ and $C=200\,000$, the numerical solution gives $E_1=118.991$ and $I_1=0.0996$, whereas the analytical calculation yields $E_1=118.97$ and $I_1=0.0989$. Using these values, we find $v_c=\Delta E/\Delta I=0.26$ (numerical) and 0.30 (analytical). (Recall that the impulse integral is identically zero for the no-vortex solution.) This reasonable agreement suggests that our Thomas-Fermi approximation should enable analytical calculations of 10–20 % accuracy for vortex critical velocities in inhomogeneous BEC, in the large-C limit, provided the calculational model and experiment have the same spatial symmetries.

Finally, in the spirit of Sec. III, we have also tested the effect of modifying the TF envelope by a regulated expression [simply changing the n^2/R^2 in Eq. (4.1) to $n^2/(R^2+n^2/\mu)$] and extending the integral over the entire disk. Doing so for $C=200\,000$, we find that the analytical estimate for v_c decreases modestly, from 0.30 to 0.29 in h.o.u., while E_1 and I_1 are both slightly closer to their respective values in the numerical integration.

V. CONCLUSION

In this paper, we report an analytical calculation of the critical velocity for the creation of a pair of counter-rotating vortices in harmonically trapped, dilute Bose condensates. Excitation of vortices may set the lowest limit for a superfluid critical velocity in BEC. For the analytical calculation presented here, we used a Thomas-Fermi approximation to the standard mean-field theory [20] in which we included the leading kinetic effects of the bulk fluid motions. This approximation is appropriate to the limit of large interparticle interactions, and thus is relevant to most current experiments. The approximation consists of two steps, described above in Sec. II: (i) solve a continuity equation for the phase of the condensate wave function, and then (ii) insert this solution for the phase in an equation of hydrostatic equilibrium and solve for the condensate envelope function, ignoring the second-order spatial derivative of the envelope function. In essence, we include contributions to the condensate kinetic energy that come from gradients of the phase (a signature of vortices), but assume that vortex-induced gradients in the condensate envelope function are negligible. Alternatively, one can think of this Thomas-Fermi approximation as equivalent to treating BEC like a fluid with an exotic equation of state, with the leading vortex kinetic energy terms acting as an effective potential. Once the envelope function is determined, we compute approximately the energy and integrated linear momentum (or impulse) of the vortex pair, and determine the critical velocity from the Landau criterion, $v_c = \min(\text{energy/impulse})$.

As described in Sec. III, we find rough quantitative agreement between our analytical calculation and the most recent BEC excitation critical velocity measurements by Ketterle and co-workers at MIT [7,8]. Also, as shown in Sec. IV, our analytical calculation of the critical velocity is in reasonable agreement with numerical calculations based on the full Gross-Pitaevskii equation *for a vortex with high symmetry*: specifically, a single vortex in the center of a two-dimensional harmonic trap.

We note two important issues that are not included in the analysis presented in this paper: (i) the breakdown of the hydrodynamic approximation at the edges of the trapped condensate, and (ii) vortex nucleation in the presence of nominally smooth trapping and perturbing potentials. Further theoretical and experimental work is necessary on these problems.

We conclude by emphasizing that the analytical method presented in this paper is straightforward to implement. In addition to enabling an analytical calculation of vortex critical velocity (for more details, see Ref. [22]), this method should have other applications in the physics of trapped BEC.

ACKNOWLEDGMENTS

We thank E. Timmermans for helpful discussions. This research was supported in part by Research Corporation Cottrell Science Award No. CC3943 and in part by the National Science Foundation under Grant Nos. PHY 94-07194 and EPS-9874764 and also by the National Science Foundation through a grant for the Institute for Theoretical Atomic and Molecular Physics at Harvard University and the Smithsonian Astrophysical Observatory.

[1] See, for example, I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965).
[2] A nice discussion of the breakdown of superfluidity in liquid ^{4}He, including many important references, is given in R. K. Pathria, *Statistical Mechanics* (Pergamon, Oxford, 1972).
[3] R. P. Feynman, in *Progress in Low Temperature Physics* (North-Holland, Amsterdam, 1955), Vol. 1, p. 17.
[4] A. Amar, Y. Sasaki, R. L. Lozes, J. C. Davis, and R. E. Packard, Phys. Rev. Lett. **68**, 2624 (1992); O. Avenel and E. Varoquaux, Phys. Rev. Lett. **55**, 2704 (1985).
[5] P. W. Anderson, Rev. Mod. Phys. **38**, 298 (1966); E. R. Huggins, Phys. Rev. A **1**, 332 (1970); L. J. Campbell, J. Low Temp. Phys. **8**, 105 (1972).
[6] C. Raman, M. Köhl, R. Onofrio, D. S. Durfee, C. E. Kuklewicz, Z. Hadzibabic, and W. Ketterle, Phys. Rev. Lett. **83**, 2502 (1999).
[7] R. Onofrio, C. Raman, J. M. Vogels, J. Abo-Shaeer, A. P. Chikkatur, and W. Ketterle, e-print cond-mat/0006111; Phys. Rev. Lett. **85**, 2228 (2000).
[8] C. Raman, R. Onofrio, J. M. Vogels, J. R. Abo-Shaeer, and W. Ketterle, e-print cond-mat/0008423.
[9] A. L. Fetter, Phys. Rev. Lett. **10**, 507 (1963); M. P. Kawatra and R. K. Pathria, Phys. Rev. **151**, 132 (1966).
[10] T. Frisch, Y. Pomeau, and S. Rica, Phys. Rev. Lett. **69**, 1644 (1992).
[11] C. Huepe and M.-E. Brachet, C. R. Acad. Sci. Paris **325**, 195 (1997).
[12] B. Jackson, J. F. McCann, and C. S. Adams, Phys. Rev. Lett. **80**, 3903 (1998).
[13] T. Winiecki, J. F. McCann, and C. S. Adams, Phys. Rev. Lett. **82**, 5186 (1999).
[14] B. Jackson, J.-F. McCann, and C. S. Adams, Phys. Rev. A **61**, 051603(R) (2000).
[15] C. Nore, C. Huepe, and M.-E. Brachet, Phys. Rev. Lett. **84**, 2191 (2000).
[16] T. Winiecki, B. Jackson, J. F. McCann, and C. S. Adams, e-print cond-mat/0004430.
[17] V. L. Ginzburg and L. P. Pitaevskii, Sov. Phys. JETP **7**, 858 (1958); E. P. Gross, J. Math. Phys. **4**, 195 (1963).
[18] J. S. Stiessberger and W. Zwerger, e-print cond-mat/0006419.
[19] P. O. Fedichev and G. V. Shlyapnikov, e-print cond-mat/0004039.
[20] See the discussion in the review article F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[21] A. L. Fetter, Phys. Rev. **138**, A429 (1965).
[22] M. Crescimanno and R. Walsworth (unpublished).
[23] A. Svidzinski and A. Fetter, e-print cond-mat/9803181.